Iterative finite element solution of multiple-scattering problems at high frequencies

C. Geuzaine[1], R. V. Sabariego[1], P. Dular[1,2] and X. Antoine[3]

[1]University of Liége, Dept. of Electrical Engineering and Computer Science, B-4000 Liége, Belgium
[2] Fonds de la Recherche Scientifique, F.R.S.-FNRS, Belgium
[3] University Henri Poincaré Nancy, Dept. of Mathematics, F-54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail: cgeuzaine@ulg.ac.be

***Abstract* — We present a multiple-scattering solver for non-convex geometries such as those obtained as the union of a finite number of convex surfaces. The algorithm is a reformulation, using finite elements, of the integral equation technique proposed in [1]. It is based on an iterative solution of the scattering problem, where each iteration leads to the resolution of a single scattering problem. At high frequencies the solution of each single-scattering problem can be greatly accelerated thanks to the *phase reduction* technique.**

I. INTRODUCTION

Solving multiple-scattering problems at high frequencies is a challenging problem, especially when the wavelength is significantly smaller than the size of the scattering obstacles.

For non-convex geometries obtained as the union of a finite number of convex surfaces, an efficient algorithm was proposed in [1] based on three main elements: 1) an iteratively computable Neumann series for the currents induced on the scattering surfaces, which accounts rigorously for multiple scattering; 2) a generalized ansatz that allows for *a priori* determination of the highly oscillatory phase of the currents in each term of the series; and 3) use of the single-scattering boundary integral solver from [2] for the efficient evaluation of each one of the terms in this series.

In this paper we present a reformulation of this algorithm using a finite element approach, which requires a fundamental rethinking of steps 2) and 3) since the fields are to be computed in the volume instead of only on the boundary of the scatterers. This new finite element approach exhibits many interesting features, amongst which possible extensions to non-homogeneous media and more complex geometries. Also, the proposed finite element formulation uses standard basis functions and can thus be easily implemented in existing finite element codes.

II. MULTIPLE-SCATTERING ITERATIONS

We investigate the numerical solution of the time-harmonic acoustic scattering problem of a plane wave $u^{\text{inc}}(\boldsymbol{x}) = e^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}$, $|\boldsymbol{\alpha}| = 1$, by a collection of impenetrable obstacles $\Omega_i^- \subset \mathbb{R}^2$, $i = 1, \dots, N$, with closed boundaries Γ_i (a TE-electromagnetic problem). Setting $\Omega^- = \cup_{i=1}^N \Omega_i^-$, $\Gamma = \cup_{i=1}^N \Gamma_i$ and $\Omega^+ = \mathbb{R}^2 \backslash \overline{\Omega^-}$, the boundary value problem reads:

$$\begin{aligned} \Delta u + k^2 u = 0 \quad &\text{in } \Omega^+, \\ u = -u^{\text{inc}} \quad &\text{on } \Gamma, \\ \lim_{|\boldsymbol{x}|\to+\infty} |\boldsymbol{x}|(\nabla u \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - iku) = 0&. \end{aligned} \tag{1}$$

This work was supported in by the Belgian Science Policy under grand IAP P6/21 and by NSF under grant DMS-0609824.

Instead of trying to solve (1) directly, we look for the solution in terms of the series $u = \sum_{m=1}^{\infty} \sum_{i=1}^{N} u_i^{(m)}$, where $u_i^{(m)}$ is the solution of the problem:

$$\begin{aligned} \Delta u_i^{(m)} + k^2 u_i^{(m)} = 0 \quad &\text{in } \mathbb{R}^2 \backslash \overline{\Omega_i^-}, \\ u_i^{(m)} = s^{(m)} - r_i^{(m)} \quad &\text{on } \Gamma_i, \\ \lim_{|\boldsymbol{x}|\to+\infty} |\boldsymbol{x}|(\nabla u_i^{(m)} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - iku_i^{(m)}) = 0&, \end{aligned} \tag{2}$$

with

$$s^{(m)} = \begin{cases} -u^{\text{inc}} & \text{for } m = 1, \\ 0 & \text{for } m > 1. \end{cases} \tag{3}$$

and

$$r_i^{(m)} = \sum_{n=1}^{m-1} \sum_{\substack{j=1 \\ j\neq i}}^{N} u_j^{(n)} - \sum_{j=1}^{i-1} u_j^{(m)}. \tag{4}$$

In other words, we perform a Gauss-Seidel-type iteration where at each step we solve a scattering problem around the single obstacle Ω_i^-, using the fields scattered from the other obstacles as boundary condition [3].

As each correction $u_i^{(m)}$ can be interpreted as the correction introduced by the m-th wave reflection [1], the iteration can be stopped when the norm of all corrections at step m is smaller than a prescribed tolerance.

III. FINITE-ELEMENT SOLUTION

Any standard finite element code can be used to solve (2). We use the approach proposed in [4], where each correction $u_i^{(m)}$ is decomposed into a source and a reaction part, leading to the implementation of the Dirichlet boundary condition on Γ_i using a volume source in the weak formulation, on a layer of elements connected to Γ_i.

If the obstacles Ω_i^- are convex each step in the iterative process can be accelerated by using the phase reduction (PR) procedure proposed in [5]. Indeed, if Ω_i^- is convex then (2) amounts to solving a *single scattering* problem in $\mathbb{R}^2 \backslash \overline{\Omega_i^-}$. The PR procedure approximates the phase of the solution of the single scattering problem and replaces the original unknown of the scattering problem with a slowly oscillatory amplitude, which can be represented on a coarse grid.

IV. NUMERICAL EXAMPLE

We consider the scattering of a plane wave e^{ikx} on an array of 2×2 circular cylinders of radius $R = 1$, separated by a distance $d = R$. We use a Bayliss-Gunzburger-Turkel-like radiation condition to truncate the infinite domain. Figure 1 shows the corrections $u_i^{(m)}$ for $m = 1, 11, 21$ and

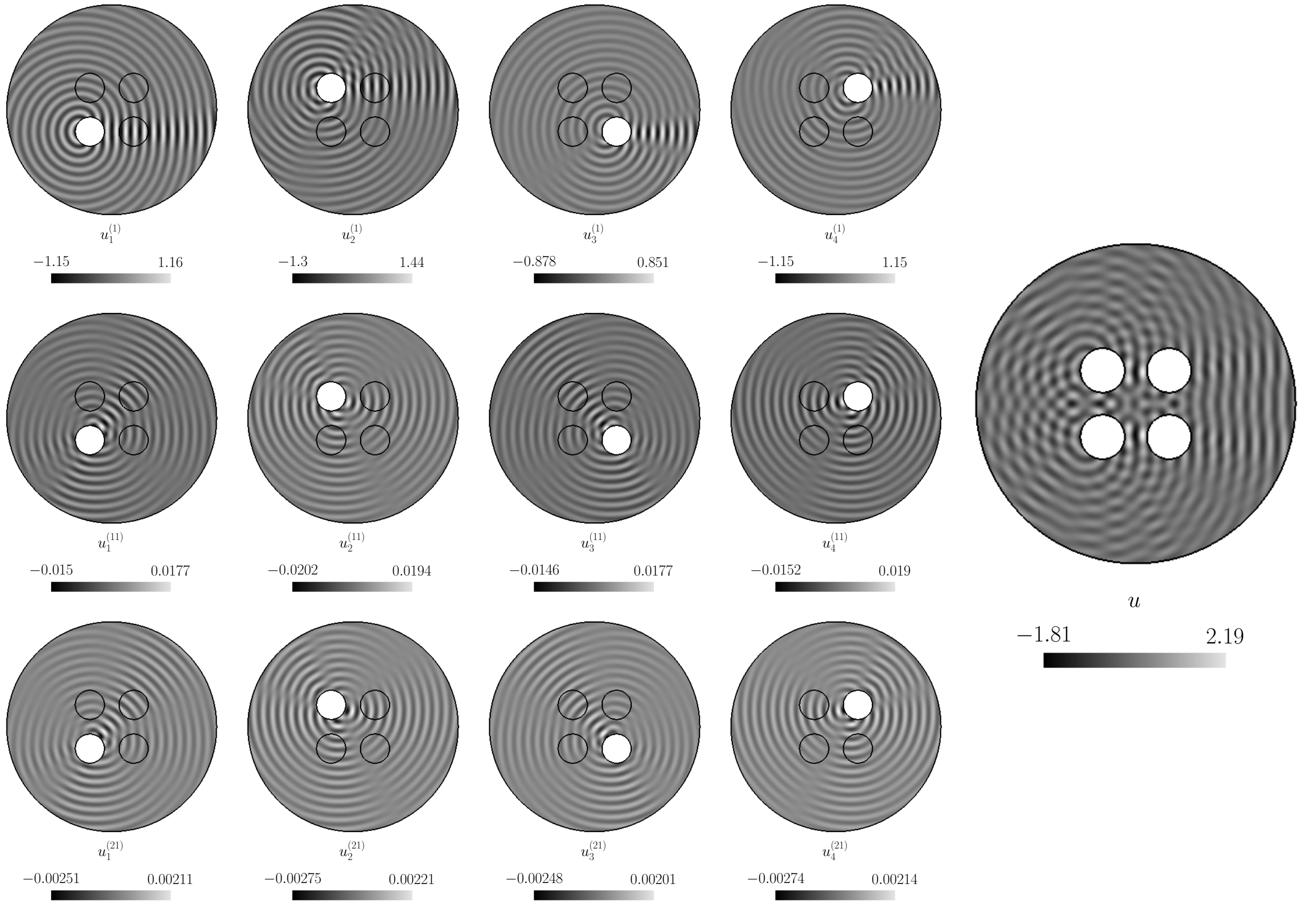

Fig. 1. Iterative solution around four circular cylinders (radius $R = 1$) for an incident plane wave arriving from the left, with wavenumber $k = 10$. Left: Real part of the corrections $u_i^{(m)}$ for $m = 1, 11, 21$ (top to bottom) and $i = 1, \ldots, 4$ (left to right). Right: Real part of the final solution.

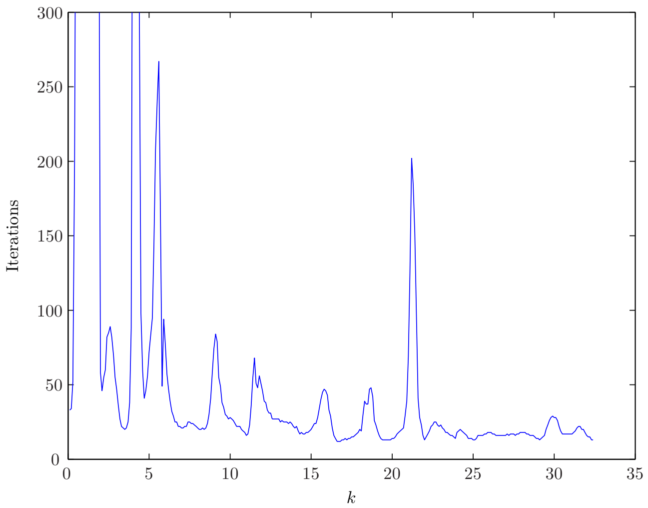

Fig. 2. Number of iterations versus wavenumber k, for a tolerance on the L^2 norm of correction equal to 10^{-3}.

$i = 1, \ldots, 4$ for $k = 10$, as well as the final solution. Of particular notice is how each correction is clearly the solution of a single scattering problem.

Figure 2 shows the convergence of the iterative process for wavenumbers $k = 0.1$ to $k = 50$. At low frequencies the algorithm diverges for some values of the wavenumber. At higher frequencies the convergence improves and the convergence rates are in agreement with the theoretical rates derived in [6] in the asymptotic (infinite frequency) case.

REFERENCES

[1] C. Geuzaine, O. Bruno, and F. Reitich. On the $\mathcal{O}(1)$ solution of multiple-scattering problems. *IEEE Transactions on Magnetics*, 41(5):1488–1491, 2005.

[2] O. Bruno, C. Geuzaine, J. Monro, Jr., and F. Reitich. Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case. *Philosophical Transactions of the Royal Society (Series A: Mathematical, Physical and Engineering Sciences)*, 362(1816):629–645, 2004.

[3] C. T. Tai. An iterative method of solving a system of linear equations and its physical interpretation from the point of view of scattering theory. *IEEE Transactions on Antennas and Propagation*, pages 713–714, September 1970.

[4] P. Dular, R. V. Sabariego, M. V. Ferreira da Luz, P. Kuo-Peng, and L. Krähenbühl. Perturbation finite-element method for magnetic circuits. *IET Science, Measurement & Technology*, 2(6):440–446, 2008.

[5] X. Antoine and C. Geuzaine. Phase reduction models for improving the accuracy of the finite element solution of time-harmonic scattering problems I: General approach and low-order models. *Journal of Computational Physics*, In Press, Available online 21 January 2009.

[6] F. Ecevit and F. Reitich. Analysis of multiple scattering iterations for high-frequency scattering problems. I: the two-dimensional case. *Preprint, School of Mathematics, University of Minnesota*, 2006.